Lie Algebras and Applications

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Dedicated to F. Lachello's Lecture Notes in Physics

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1 Basic Concepts

1.1 Definitions

Definition 1. (Commutator) *The commutator (or bracket) of X and Y is defined as*

$$[X, Y] = XY - YX \tag{1}$$

It satisfies the relations

$$[X, X] = 0; \quad [X, Y] = -[Y, X] \tag{2}$$

Definition 2. (Lie Algebras) *A set of elements $X_\alpha(\alpha = 1, 2, \ldots, r)$ is said to form a Lie algebra G, written as $X_\alpha \in G$, if the following axioms are satisfied:*

- *Axiom 1. The commutator of any two elements is a linear combination of the elements in the Lie algebra,*

$$[X_\rho, X_\sigma] = \sum_\tau c_{\rho\sigma}^\tau X_\tau \tag{3}$$

- *Axiom 2. The double commutators of three elements satisfy the Jacobi identity:*

$$[X_\rho, [X_\sigma, X_\tau]] + [X_\sigma, [X_\tau, X_\rho]] + [X_\tau, [X_\rho, X_\sigma]] = 0 \tag{4}$$

The coeeficients $c_{\rho\sigma}^\tau$ are called *Lie structure constants*.

Remark 3. Unless otherwise specified, a summation convention over repeated indices will be used

$$c_{\rho\sigma}^\tau X_\tau \equiv \sum_\tau c_{\rho\sigma}^\tau X_\tau \tag{5}$$

In a vector space, we can define the Lie algbra of it by the following definition:

Definition 4. (Lie algebras over fields) *A vector space L over a number field F, with an operation $L \times L \to L$, denoted $[X, Y]$ and called the commutator of X and Y, is called a Lie algebra over F if the following axioms are satisfied:*

(1) The operation is bilinear;

(2) $[X, X] = 0$ *for all* $X \in L$*;*

(3) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, $(\forall X, Y, Z \in L)$

Remark 5. In definition 4, the meaning of commutator is totally defined in three axioms, not always in the form of equation 1. However, we can use axioms (2) and (3) to get $[X, Y] = -[Y, X]$. Indeed, we just can consider $[X + Y, X + Y] = 0$.

Example 6.

(1) The algebra $[X_i, X_j] = X_k$, where $1 \leqslant i \neq j \neq k \leqslant 3$ is a real Lie algebra[1] with three elements. This is the angular momentum algebra in three dimension, so(3).

1. A real Lie algebra if the basic field F is $\mathbb{R}$, and called complex if F is $\mathbb{C}$. Real Lie algebras have real structure constants, while complex lie algebras have real or complex structure constants.

(2) The algebra $[X_1, X_2] = X_3$, $[X_2, X_3] = -X_1$, $[X_3, X_1] = X_2$ is also a real liealgebra with three elements. This is the Lorentz algebra in $2+1$ dimensions, so$(2,1)$.

1.2 Change of Basis

It's possible to change the basis $X'_\sigma = a^\rho_\sigma X_\rho$ where a^ρ_σ is non-singular. The new commutation relations of the algebra are

$$[X'_\rho, X'_\sigma] = c'^{\;\tau}_{\rho\sigma} X'_\tau \tag{6}$$

In general a^σ_ρ can be a complex number (complex extension) or a real number.

Remark 7. (Isomorphim) Lie algebras that have the same comutation relations up to a change of basis are called isomorphic. Isomorphism of two algebras are denoted by the symbol $\sim$.

Example 8. The Lie algebras so(3) and su(2) are isomorphic.

1.2.1 Complex Extensions

The change basis can be complex.

Example 9. The real Lie algebra so$(2,1)$ and so(3) have the same complex extension.

Consider so$(2,1)$ and making the change of basis,

$$Y_1 = X_1, Y_2 = -i\,X_2, Y_3 = -i\,X_3$$

Then, we obtain $[Y_i, Y_j] = Y_k$, where $i \neq j, j \neq k, k \neq i$, i.e. this is so$(3)$.

1.3 Lie subalgebras

Lie subalgebra is a subalgebra of Lie algebra, with closed linear combinition representation.

1.3.1 Abelian Algebras

Definition 10. (Abelian Algebras) *An Abelian algebra, $\mathcal{A}$, is an algebra for which all elements commute,*

$$[X_\rho, X_\sigma] = 0, \forall X_\rho, X_\sigma \in \mathcal{A}$$

Example 11. The algebra $t(2)$ with comutation relations

$$[X_1, X_2] = 0, [X_1, X_1] = 0, [X_2, X_2] = 0$$

is Abelian.

1.3.2 Direct Sum

This is actually an inner direct summation, which is used to decompose an algebra.

Consider two algebras $g_1 \ni X_\alpha$, $g_2 \ni X_\beta$, satisfying

$$\begin{aligned} [X_\rho, X_\sigma] &= c^\tau_{\rho\sigma} X_\tau \\ [Y_\rho, Y_\sigma] &= c'^{\tau}_{\rho\sigma} Y_\tau \\ [X_\rho, Y_\sigma] &= 0 \end{aligned} \tag{7}$$

This commuting property is denoted by $g_1 \cap g_2 = 0$. Then the set of elements X_α, Y_β form an algebra g, called the direct sum, $g = g_1 \oplus g_2$.

Somtimes, it is possible to rewrite a Lie algebra as a direct sum of other algebras, usually it must do a change of basis first. Consider following example, so(4) $\ni X_1, X_2, X_3, Y_1, Y_2, Y_3$, satisfying commutation relations:

$$\begin{array}{ccc} [X_1, X_2] = X_3; & [X_2, X_3] = X_1; & [X_3, X_1] = X_2 \\ [Y_1, Y_2] = X_3; & [Y_2, Y_3] = X_1; & [Y_3, Y_1] = X_2 \\ [X_1, Y_1] = 0; & [X_2, Y_2] = 0; & [X_3, Y_3] = 0 \\ [X_1, Y_2] = Y_3; & [X_1, Y_3] = -Y_2; & [X_2, Y_1] = -Y_3 \\ [X_2, Y_3] = Y_1; & [X_3, Y_1] = Y_2; & [X_3, Y_2] = -Y_1 \end{array}$$

By changing basis,

$$J_i = \frac{X_i + Y_i}{2}, K_i = \frac{X_i - Y_i}{2} \quad (i = 1, 2, 3)$$

The the algebra can be brought to the form:

$$\begin{array}{ccc} [J_1, J_2] = J_3; & [J_2, J_3] = J_1; & [J_3, J_1] = J_2 \\ [K_1, K_2] = K_3; & [K_2, K_3] = K_1; & [K_3, K_1] = K_2 \\ & [J_i, K_j] = 0; & \end{array}$$

i.e. we have

$$\mathrm{so}(4) \sim \mathrm{so}(3) \oplus \mathrm{so}(3) \sim \mathrm{su}(2) \oplus \mathrm{su}(2) \sim \mathrm{sp}(2) \oplus \mathrm{sp}(2)$$

What's more, so(4) and so(3, 1) have the same complex extension, then they can be also viewed as same split.

1.3.3 Ideals (Invariant Subalgebras)

Definition 12. *Consider an algebra g and its subalgebra g',$X_\alpha \in g$, $Y_\beta \in g'$,$g \supset g'$. Since g' is a subalgebra, it satisfies*

$$[Y_\rho, Y_\sigma] = c^\tau_{\rho\sigma}{}' Y_\tau$$

what's more, if in addition,

$$[Y_\rho, X_\sigma] = c^\tau_{\rho\sigma} Y_\tau$$

Then g' is called an invariant subalgebra.

1.4 Semisimple Algebras

An algebra which has no Abelian ideals is called semisimple.

Example 13. the algebra so(3) is semisimple, while algebra $e(2)$

$$[X_1, X_2] = X_3, [X_1, X_3] = -X_2, [X_2, X_3] = 0$$

is non-semisimple.

1.4.1 Semidirect Sum

Consider two non-commutative algebras, $X_\alpha \in g_1, Y_\beta \in g_2$, satisfying

$$[X_\rho, X_\sigma] = c_{\rho\sigma}^\tau X_\tau, [Y_\rho, Y_\sigma] = {c'_{\rho\sigma}}^\tau Y_\tau$$

If g_2 is an ivariant subalgebra (ideal) of $g_1, i.e.$

$$[X_\rho, Y_\sigma] = {c''_{\rho\sigma}}^\tau Y_\tau$$

Then the algebra g,

$$g = g_1 \oplus_s g_2$$

is called the semidirect sum of g_1 and g_2.

An example is $e(2) = \mathrm{so}(2) \oplus_s t(2)$.

1.4.2 Killing Form

With the Lie structure constants one can form a *tensor*, called *metric tensor* or *Killing form.*

$$g_{\sigma\lambda} = g_{\lambda\sigma} = c_{\sigma\rho}^\tau c_{\lambda\tau}^\rho \tag{8}$$

Theorem 14. *A Lie algebra g is a semisimple if, and only if,*

$$\det(g_{\sigma\lambda}) \neq 0 \tag{9}$$

Remark 15. the killing form is a symmetric matrix, and each component is the tensor product of structure constants.

usually, we write $g^{\sigma\lambda}$ as the inverse of tensor $g_{\sigma\lambda}$.

Example 16. The algebra so(3) is semisimple.

The metric tensor of so(3) is

$$g_{\sigma\lambda} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Hence, so(3) is semisimple.

Example 17. The algebra so(2,1) is semisimple.

The metric tensor of so(2, 1) is

$$g_{\sigma\lambda} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Hence, so(2, 1) is semisimple.

1.4.3 Compact and Non-Compact Algebras

A real semisimple Lie algebra is compact if its metric tensor is negative definite.

Example 18. so(3) is compact since the diagonal form all negative, while so(2, 1) is non-compact,since there exists positive element in diagonal form.

1.5 Derivations

Definition 19. (Derivations) *Starting with a Lie algebra, g, with elements X_ρ, it's possible to construct other algebras, called the derivations and denoted by* $\mathrm{Der}\, g$, *by taking commutators*

$$\begin{aligned} \mathrm{Der}\, g = g^{(1)} &= [g, g] \\ \mathrm{Der}^2 g = g^{(2)} &= [g^{(1)}, g^{(1)}] \\ &\cdots \end{aligned}$$

If for some positive k, s.t.

$$\mathrm{Der}^k g = 0$$

The algebra g is called *solvable*.

Example 20. e(2) is solvable, since $\mathrm{Der}^2\, e(2) = 0$.

1.6 Nilponent Algebras

Starting with Lie algebra g with elements X_ρ, it's possible to construct powers of g as

$$\begin{aligned} g^2 &= g^{(1)} = [g, g] \\ g^3 &= [g, g^2] \\ &\cdots \end{aligned}$$

If for some positive k, $g^k = 0$, then the algebra is called nilponent.

Example 21. e(2) is not nilponent, since $g^i = \{X_2, X_3\}$, if $i \geqslant 2$.

1.7 Invariant Casimir Operators

These operators play a central role in application, which are named after the Dutch physicist Casimir.

Definition 22. (Invariant Casimir Operators) *An operator, C, that commutes with all the elements of Lie algebra g,*

$$[C, X_\tau] = 0, \forall X_\tau \in g \tag{10}$$

is called an invariant Casimir operator.

Remark 23. Casimir operators can be linear, quadratic, ... in the elements X_τ. A Casimir operator is called of order p is it contains products of elements X_τ.

$$C_p = \sum_{\alpha_1, \ldots, \alpha_p} f^{\alpha_1 \alpha_2 \ldots \alpha_p} X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_p}$$

Remark 24. The quadratic Casimir operator of a *semisimple algebra* can be simply constructed from the metric tensor

$$C_2 = g^{\rho\sigma} X_\rho X_\sigma = g_{\rho\sigma} X^\rho X^\sigma \equiv C \tag{11}$$

Higher order Casimir operators can be constructed in a similar fashion[2].

Example 25. For the algebra so(3), the inverse of metric tensor is

$$g^{\sigma\lambda} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

giving

$$C = -\frac{1}{2}(X_1^2 + X_2^2 + X_3^2)$$

For the algebra so$(2, 1)$,

$$C = -\frac{1}{2}(X_1^2 - X_2^2 - X_3^2)$$

1.7.1 Invariant Operators for Non-semisimple Algebras

For non-semisimple lie algebra, Casimir operators cannot be simply constructed.

Example 26. e(2) has an invariant operator $C = X_2^2 + X_3^2$.

1.8 Structure of Lie Algebra

1.8.1 Algebras with one element

In this case there is only one element X, and one posibility,

$$(a) \quad [X, X] = 0 \tag{12}$$

This algebra is Abelian.

2. It has same meaning with "way" or "method".

The algebras so(2) $\sim u(1)$ are examples of case 1(a)

1.8.2 Algebras with two elements

In this case, there are two elements X_1, X_2, and two posibilities:

$$(a) \quad [X_1, X_2] = 0 \tag{13}$$

and

$$(b) \quad [X_1, X_2] = X_1 \tag{14}$$

In case (a), the algebra is Abelian. In case (b), X_1 is an Abelian ideal.

The translation algebra $t(2)$ is an example of case 2(a).

1.8.3 Algebras with three elements

For $r = 3$, there are three elements X_1, X_2, X_3, and four posibilities:

$$(a) \quad [X_1, X_2] = [X_2, X_3] = [X_3, X_1] = 0 \tag{15}$$

$$\begin{array}{ll} (b) & [X_1, X_2] = X_3; [X_1, X_3] = [X_2, X_3] = 0 \\ \text{or} & [X_1, X_3] = X_2; [X_1, X_2] = [X_2, X_3] = 0 \end{array} \tag{16}$$

$$(c)\, [X_1, X_2] = 0; [X_3, X_1] = \alpha X_1 + \beta X_2; [X_3, X_2] = \gamma X_1 + \delta X_2 \tag{17}$$

with the matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ is non-singular.

$$\begin{array}{ll} (d) & [X_1, X_2] = X_3; [X_2, X_3] = X_1; [X_3, X_1] = X_2 \\ \text{or} & [X_1, X_2] = X_3; [X_2, X_3] = -X_1; [X_3, X_1] = X_2 \end{array} \tag{18}$$

In case (a), the algebra is Abelian, an example is $t(3)$. $e(2)$ is an example of $3(c)$, with $\alpha = \delta = 0$, $\beta = 1, \gamma = -1$. so(3) and so$(2,1)$ is examples of $3(d)$.

This procedure becomes very cumbersome as the number of elements in the algebra increases.